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# On the spin-up of an electrically conducting fluid
# Part 1. The unsteady hydromagnetic Ekman–Hartmann boundary-layer problem

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The prototype spin-up problem between infinite flat plates treated by Greenspan & Howard (1963) is extended to include the presence of an imposed axial magnetic field. The fluid is homogeneous, viscous, and electrically conducting. The resulting boundary initial-value problem is solved to first order in Rossby number by Laplace transform techniques. In spite of the linearization the complete hydromagnetic interaction is preserved: currents affect the flow and the flow simultaneously distorts the field. In part 1, we analyze the impulsively started time dependent approach to a final steady Ekman–Hartmann boundary layer on a single insulating flat plate. The transient is found to consist of two diffusively growing boundary layers, inertial oscillations, and a weak Alfvén wave front. In part 2, these one plate results are utilized in discussing spin-up between two infinite flat insulating plates. Two distinct and important hydromagnetic spin-up mechanisms are elucidated. In all cases, the spin-up time is found to be shorter than in the corresponding non-magnetic problem.

## 1. Introduction

A conceptually simple problem, which has provided much valuable insight into the dynamics of homogeneous rotating fluids in recent years, is the so-called spin-up problem. In the simplest case it concerns the manner in which a contained fluid adjusts from one state of rigid body rotation to another (with the rotation axis fixed in direction). It has now been shown (Greenspan 1964, 1965, 1968; Greenspan & Weinbaum 1965) that the essential dynamics for the general spin-up problem in an arbitrary container with finite (e.g. non-linear) change in angular speed can largely be understood by solving the much simpler prototype problem treated by Greenspan & Howard (1963). In their spin-up problem, the fluid is 'contained' between two infinite flat parallel plates perpendicular to the rotation axis, and only an infinitesimal change in rotation speed is allowed so that the mathematical problem becomes linear.

Clearly, extensions of this work are required before it can be applied directly to geophysical or astrophysical problems of interest. One such extension of importance is to consider the effects of a stable density stratification as has been done by Pedlosky (1967). In the present paper we study the prototype *hydromagnetic* spin-up problem for a homogeneous fluid. The detailed motivation for, and application of, this work will not be given here, but it is worth mentioning that the slow westward drift of the geomagnetic field and the 'solar spin-down problem' (e.g. Howard, Moore & Spiegel 1967) both suggest the relevance of 'spin-up problems' for electrically conducting fluids. Apart from this motivation, it will be seen that the idealized problem treated here is sufficiently rich in phenomena that it deserves attention for its own sake.

More specifically, the ultimate aim here is to understand how a homogeneous, viscous electrically conducting fluid contained between two infinite, flat, parallel, insulating plates adjusts in time from one rigid body rotation to a slightly different collinear one, when an applied uniform magnetic field acts parallel to the rotation axis. It is believed that this highly idealized problem embraces many (though certainly not all) of the fundamental processes which must be crucial in the more complicated problems of physical interest. Relying heavily on the now well understood non-magnetic problem (Greenspan & Howard), it seems clear that three successive studies are required to 'solve' the stated problem. First, a knowledge of the hydromagnetic analogue of a steady Ekman boundary layer is required. Secondly, the establishment and transient dynamics of such a layer must be exposed. Finally, two such layers must be allowed to interact with each other and a non-dissipative core flow to produce spin-up. The first part of this programme is the subject of a recent paper by Gilman & Benton (1968), which will be briefly summarized next. The second and third steps above are the subjects of parts 1 and 2 of this paper, respectively.

One of the two important hydromagnetic spin-up mechanisms, that will be elaborated on in part 2, owes its existence primarily to the interesting features of the *steady* Ekman–Hartmann boundary layer, discussed by Gilman & Benton. In their work, a single infinite flat insulating plate rotates with angular speed $\Omega_0$, and the viscous electrically conducting fluid far from it rotates with slightly different speed $\Omega_1 = \Omega_0(1+\epsilon)$. The unperturbed magnetic field is uniform and perpendicular to the plates. In the limits of vanishing and infinite magnetic field, the boundary layer reduces, respectively, to the classical Ekman layer and a rotating Hartmann layer. An MHD extension of von Kármán radial similarity, together with an expansion in powers of the Rossby number $\epsilon$, leads to the exact axisymmetric solution. To first order in $\epsilon$, the impressed magnetic field has two important consequences. First, because of its Maxwell tension, the Ekman pumping by unbalanced centrifugal forces near the boundary produces a weaker radial motion than in the non-magnetic case; consequently, indirectly through mass continuity, Ekman suction (or blowing) is inhibited by the hydromagnetic body force (as it is also directly inhibited by buoyancy forces when the density is stably stratified, Barcilon & Pedlosky 1967). Secondly, the imposed vertical shear of the tangential velocity tips the applied axial magnetic field lines partially into the azimuthal direction; thus, an axial electric current (called the 'Hartmann current') is induced, and it persists outside the boundary layer. It will be shown (in part 2) how this current can substitute for the inhibited Ekman suction velocity in producing rapid hydromagnetic spin-up by an accelerating tangential electromagnetic body force.

In the present paper, part 1, we examine the impulsively started initial value problem for the transient approach to steady, linear, Ekman–Hartmann flow. As will be seen, this represents a non-trivial yet mathematically tractable extension of ordinary Ekman theory with many fascinating features. Indeed, there are two diffusively growing boundary layers, highly persistent inertial oscillations, and an Alfvén wave front, whose properties and interactions are studied by Laplace transform techniques. The mathematical formulation and simplification are given in § 2 and the exact Laplace transform solution in § 3. Section 4 deals with some limiting cases of interest for which the exact inversions are available, in particular, ordinary non-magnetic flow, the early and late time behaviour in general, and flow at zero magnetic Prandtl number. The physically interesting case of small non-vanishing magnetic Prandtl number is treated approximately in § 5, and the results are summarized in § 6. Part 2 of the paper, entitled *Hydromagnetic spin-up between infinite flat, insulating plates*, utilizes the results of part 1 to discuss both qualitatively and quantitatively how the spin-up process occurs and presents formulae for the spin-up time.

## 2. Mathematical formulation

We consider the following problem. Prior to time $t = 0$, a homogeneous, viscous, electrically conducting fluid and its boundary, the insulating half-space $z < 0$, are in rigid body co-rotation at angular speed $\Omega$ about the $z$ axis. A uniform magnetic field, of strength $B_0$ is imposed parallel to the $z$ axis. No electric currents flow in the basic state. At time $t = 0$, the boundary angular speed is impulsively accelerated to the value $\Omega(1+\epsilon)$, and the applied field is not changed. The parameter $\epsilon$, the Rossby number, is of small magnitude compared to 1; for purposes of discussion, it is regarded as positive, so that the boundary has been spun-up; but, clearly, to first order in $\epsilon$, spin-down is simply a suitable 'reflexion' of spin-up.

Mathematically, we need to solve the fundamental MHD equations, which are written in an inertial co-ordinate system and rationalized MKS units (Shercliff 1965) as:

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla(\tfrac{1}{2}\mathbf{v}^2) + (\nabla \times \mathbf{v}) \times \mathbf{v} = -\nabla\pi + \frac{1}{\rho\mu}(\nabla \times \mathbf{B}) \times \mathbf{B} - \nu\nabla \times (\nabla \times \mathbf{v}), \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \lambda\nabla \times (\nabla \times \mathbf{B}), \tag{2}$$

$$\nabla . \mathbf{v} = \nabla . \mathbf{B} = 0, \tag{3}$$

where $\mathbf{v}$, $\mathbf{B}$, $p = \rho\pi$, $\rho$, $\mu$, $\nu$, $\lambda = (\mu\sigma)^{-1}$ and $\sigma$ are respectively, velocity field, magnetic field, pressure, density, magnetic permeability, kinematic viscosity, resistivity and electrical conductivity. Suitable initial and boundary conditions, in cylindrical co-ordinates, are:

at $t = 0$: $$\mathbf{v} = r\Omega\hat{\theta}, \quad \mathbf{B} = B_0\hat{z}, \tag{4}$$

for $t > 0$: $$\left.\begin{aligned} &\text{at} \quad z = 0, \quad \mathbf{v} = r\Omega(1+\epsilon)\hat{\theta}, \quad \mathbf{B} \text{ is continuous,} \\ &\text{as} \quad z \to \infty, \quad v_r \to 0, \quad v_\theta \to r\Omega, \quad \mathbf{B} \to B_0\hat{z}. \end{aligned}\right\} \tag{5}$$

The continuity condition on $\mathbf{B}$ at $z = 0$ allows us to match the field within the fluid region to a potential field within the insulating half-space (Shercliff 1965). The reason why all perturbations in $\mathbf{B}$ must decay to zero as $z \to \infty$ will be made clear presently.

Major mathematical simplifications occur, because the exact solution is axially symmetric ($\partial/\partial\theta \equiv 0$), has known radial dependence of the von Kármán similarity type (i.e. both vector fields $\mathbf{v}$ and $\mathbf{B}$ have radial and azimuthal components proportional to $r$, and axial component independent of $r$), and can be expressed as a power series in Rossby number $\epsilon$. Expansions correct to first order in $\epsilon$, which embody these ideas and also non-dimensionalize the variables, are given for the velocity, pressure, magnetic field and electric current, in cylindrical co-ordinates $r$, $\theta$, $z$, as (cf. Gilman & Benton):

$$\mathbf{v}(r,z,t) = r\Omega\hat{\theta} + \Omega\epsilon[rU(\zeta,\tau)\hat{r} + rV(\zeta,\tau)\hat{\theta} + d_E W(\zeta,\tau)\hat{z}], \tag{6}$$

$$\pi(r,z,t) = \tfrac{1}{2}r^2\Omega^2 + \nu\Omega\epsilon Q(\zeta,\tau), \tag{7}$$

$$\mathbf{B}(r,z,t) = B_0\hat{z} + B_0\mu\sigma(\nu\Omega)^{\frac{1}{2}}\epsilon[rA(\zeta,\tau)\hat{r} + rB(\zeta,\tau)\hat{\theta} + d_E C(\zeta,\tau)\hat{z}], \tag{8}$$

$$\mathbf{j}(r,z,t) \equiv (1/\mu)\nabla\times\mathbf{B} = B_0\sigma\Omega\epsilon[rI(\zeta,\tau)\hat{r} + rJ(\zeta,\tau)\hat{\theta} + d_E K(\zeta,\tau)\hat{z}]. \tag{9}$$

In these expressions, $d_E = (\nu/\Omega)^{\frac{1}{2}}$, is the Ekman depth, $\zeta = z/d_E$ and $\tau = \Omega t$. The particular scaling for $\mathbf{B}$ and $\mathbf{j}$ is chosen to reduce the frequency with which parameters occur later on. Note that the basic unperturbed state ($\epsilon = 0$) is simply the uniform rotation (i.e. uniform vorticity) and uniform collinear magnetic field, with no electric currents, no hydromagnetic interaction, and pressure just balancing centrifugal force arising from the rotation. In what follows we study small departures from this state. Since $\mathbf{j}$ is proportional to the curl of $\mathbf{B}$, (8) and (9) combine to give

$$\mathbf{j} = B_0\sigma\Omega\epsilon\left[-r\frac{\partial B}{\partial\zeta}\hat{r} + r\frac{\partial A}{\partial\zeta}\hat{\theta} + 2d_E B\hat{z}\right]. \tag{10}$$

Thus, only radial and tangential magnetic field components correspond to currents; note also that the important axial component of current comes from an azimuthal field.

Substitution of the expansions (6)–(8) into the fundamental equations (1)–(3), and neglect of terms quadratic in $\epsilon$, leads to the following two-parameter linear partial differential equation set:

$$U_\tau - U_{\zeta\zeta} = 2V + 2\alpha^2 A_\zeta, \tag{11}$$

$$V_\tau - V_{\zeta\zeta} = -2U + 2\alpha^2 B_\zeta, \tag{12}$$

$$\delta A_\tau - A_{\zeta\zeta} = U_\zeta, \tag{13}$$

$$\delta B_\tau - B_{\zeta\zeta} = V_\zeta, \tag{14}$$

$$\delta C_\tau - C_{\zeta\zeta} = W_\zeta, \tag{15}$$

$$W_\zeta = -2U, \tag{16}$$

$$C_\zeta = -2A, \tag{17}$$

$$Q_\zeta = W_{\zeta\zeta} - W_\tau. \tag{18}$$

It is important to note that in spite of the linearization, the full hydromagnetic coupling has been preserved; currents affect the flow and simultaneously the flow distorts the field lines. The two fundamental parameters of the problem are:

$$\alpha = \frac{d_E}{\sqrt{2}\,d_H} = \left(\frac{\sigma}{2\rho\Omega}\right)^{\frac{1}{2}} B_0 = \text{magnetic interaction parameter,} \tag{19}$$

$$\delta = \frac{\nu}{\lambda} = \sigma\mu\nu = \text{magnetic Prandtl number.} \tag{20}$$

In (19), $d_H$ is the depth of an ordinary Hartmann boundary layer,

$$d_H = (\rho\nu/\sigma B_0^2)^{\frac{1}{2}},$$

so $\sqrt{2}\,\alpha$ is the ratio of Ekman to Hartmann depth. Note that this ratio is independent of kinematic viscosity, and that from its position in the equations $2\alpha^2$ measures the strength of the electromagnetic body force relative to the Coriolis force. The magnetic Prandtl number is simply the ratio of the rate of viscous to magnetic diffusion. For most geophysical, astrophysical or engineering situations of interest, $\nu \ll \lambda$, so in what follows attention is restricted to the range $\delta \ll 1$; however, as we shall soon see, the limit $\delta \to 0$ is a singular one (refer to (13)–(15)), so this limit is only taken after the full solution has been obtained. Since the relative magnetic field strength can vary considerably from one physical situation to another we attempt to obtain solutions uniformly valid for all $\alpha$.

The initial and boundary conditions now take the form,

$$\text{at} \quad \tau = 0, \quad U = V = W = A = B = C = 0 \quad \text{for} \quad \tau > 0, \tag{21}$$

$$\left.\begin{array}{ll} \text{at} \quad \zeta = 0, & V = 1, \quad U = W = A = B = C = 0, \\ \text{as} \quad \zeta \to \infty, & U, V, A, B, C \to 0. \end{array}\right\} \tag{22}$$

The boundary conditions on the velocity need no comment. Those on the magnetic field at $\zeta = 0$ follow, by continuity, from the required absence of any perturbation within the insulator (since no current flows therein, $\nabla \times \mathbf{B}$ and $\nabla . \mathbf{B}$, both vanish, and $\mathbf{B}$ must be bounded as $\zeta \to -\infty$). In the fluid, as $\zeta \to +\infty$, the perturbation magnetic field vanishes by virtue of the differential equations; no other bounded solutions exist.

Some of these conditions at $\zeta = 0$ and $\zeta = \infty$ differ from those used in the steady analysis of Gilman & Benton. This point will be clarified later. For the moment, it suffices to say that this apparent discrepancy involves a non-uniformity in the large time behaviour, which the present analysis resolves. Furthermore, the matter is largely academic, because the steady flow and electric current structure of Gilman & Benton are completely independent of the difference in these particular boundary conditions.

Equations (11)–(18) are eight equations for seven unknowns. Equation (15), being redundant, in view of (13) and (17), is disregarded. Furthermore, all variables of interest can be computed from $U$, $V$, $A$, $B$ whose equations (11)–(14) form a closed set. A considerable economy of notation is achieved by introducing complex notation for the velocity components parallel to the plates (with $F$ for 'fluid') and similarly for the field ($M$ for 'magnetic'):

$$F(\zeta, \tau) = U(\zeta, \tau) + iV(\zeta, \tau), \tag{23}$$

$$M(\zeta, \tau) = A(\zeta, \tau) + iB(\zeta, \tau). \tag{24}$$

With this notation, the problem is reduced to solving

$$F_\tau - F_{\zeta\zeta} + 2iF = 2\alpha^2 M_\zeta, \tag{25}$$

$$\delta M_\tau - M_{\zeta\zeta} = F_\zeta, \tag{26}$$

subject to

$$F(0,\tau) = i, \quad F(\zeta,0) = M(\zeta,0) = F(\infty,\tau) = M(0,\tau) = M(\infty,\tau) = 0. \tag{27}$$

The terms in (25) are, respectively, the local acceleration, viscous diffusion, Coriolis acceleration, and electromagnetic body force. Those in (26) are unsteady change in the magnetic field, magnetic or resistive diffusion, and induction of magnetic field.

Equations (25) and (26) can easily be combined into a single (in this case identical) equation for either $F$ or $M$. However, since the boundary conditions cannot easily be combined, this approach is not particularly fruitful.

## 3. The Laplace transform solution

Some care must be exercised in utilizing the Laplace transform of $F$ (or $M$), defined, for example, by

$$\bar{F}(\zeta,s) = \int_0^\infty e^{-s\tau} F(\zeta,\tau)\,d\tau, \tag{28}$$

because $F$ and $M$ are complex functions, and $s$ is also a complex variable. Specifically, whenever $\bar{U}$, $\bar{V}$, $\bar{W}$, $\bar{A}$, $\bar{B}$, are to be extracted from $\bar{F}$ or $\bar{M}$, as, for example, in

$$\bar{U}(\zeta,s) = \operatorname{Re}\bar{F}(\zeta,s),$$

$$\bar{W}(\zeta,s) = -2\operatorname{Re}\int_0^\zeta \bar{F}(\xi,s)\,d\xi,$$

then $s$ must be treated as a real variable during the operations of taking real or imaginary part (Doetsch 1961). The transformed problem is now

$$\left.\begin{aligned} &\bar{F}'' - (s+2i)\bar{F} = -2\alpha^2\bar{M}', \\ &\bar{M}'' - \delta s\bar{M} = -\bar{F}', \\ &\bar{F}(0,s) = is^{-1}, \quad \bar{F}(\infty,s) = \bar{M}(0,s) = \bar{M}(\infty,s) = 0, \end{aligned}\right\} \tag{29}$$

where primes denote differentiation with respect to $\zeta$. Clearly, for these linear equations with constant coefficients, only exponential solutions will be obtained. They will either grow without limit or decay to zero as $\zeta \to \infty$. This explains why the perturbation magnetic field actually must decay to zero as $\zeta \to \infty$; the only other physically acceptable possibility, a bounded constant, is not a solution of the equations. This fact has some important consequences.

The exact solution of (29) is:

$$\bar{F}(\zeta,s) = i\frac{k(s+2i-m^2)\,e^{-k\zeta} - m(s+2i-k^2)\,e^{-m\zeta}}{(k-m)\,s(s+2i)^{\frac{1}{2}}\,[(s+2i)^{\frac{1}{2}} + (\delta s)^{\frac{1}{2}}]}, \tag{30}$$

$$\bar{M}(\zeta,s) = i\,\frac{(s+2i)^{\frac{1}{2}}}{(k-m)\,s[(s+2i)^{\frac{1}{2}} + (\delta s)^{\frac{1}{2}}]}\,(e^{-k\zeta} - e^{-m\zeta}), \tag{31}$$

where
$$k = [n+(n^2-q^2)^{\frac{1}{2}}]^{\frac{1}{2}}, \quad \operatorname{Re} k \geqslant 0, \tag{32}$$
$$m = [n-(n^2-q^2)^{\frac{1}{2}}]^{\frac{1}{2}}, \quad \operatorname{Re} m \geqslant 0, \tag{33}$$
$$n = \tfrac{1}{2}[(1+\delta)s+2\alpha^2+2i], \tag{34}$$
$$q = (\delta s)^{\frac{1}{2}}(s+2i)^{\frac{1}{2}}, \quad \operatorname{Re} q \geqslant 0. \tag{35}$$

The transform of the axial velocity is:

$$\overline{W}(\zeta,s) = \operatorname{Re}\left\{-2i\,\frac{(k^2-m^2)-(s+2i-m^2)\,e^{-k\zeta}+(s+2i-k^2)\,e^{-m\zeta}}{(k-m)\,s(s+2i)^{\frac{1}{2}}\,[(s+2i)^{\frac{1}{2}}+(\delta s)^{\frac{1}{2}}]}\right\}, \tag{36}$$

and the real part here is taken with $s$ considered real.

It is to be noted from (30), (31) and (36) that the flow has a double layer exponential structure in $\zeta$. These layers are referred to as the $k$ and $m$ layers; their physical significance will emerge shortly. In the steady state problem of Gilman & Benton only a single Ekman–Hartmann layer exists. One problem then is to explain this seeming paradox.

The inversion of (30), (31) and (36) requires a knowledge of the location and type of singularities. A careful inspection shows that in general, the only singularities are a simple pole at $s = 0$ (which gives the steady state solution), and branch points at $s = 0$ and $s = -2i$. The functions can easily be shown to have no singularity at possible values of $s$ for which $k = m$. Also, if the branch with positive real part is chosen for the square roots which occur, then clearly $(s+2i)^{\frac{1}{2}}+(\delta s)^{\frac{1}{2}}$ does not vanish anywhere (except when $\delta = 0$ and $s = -2i$). In this last case ($\delta = 0$), and generally for $\delta \ll 1$, there is an additional singularity at $s = -2\alpha^2-2i$, whose effects are studied in §§4(iv) and 5.

## 4. Laplace inversion for some limiting cases of interest

This section is devoted to the study of some important limiting cases, for which exact results can be obtained. The special limits chosen here are selected to give overall perspective to the study.

### (i) *Ordinary hydrodynamic flow* ($\alpha = 0$)

To check that the non-magnetic case is in order, set $B_0 = 0$, which implies $\alpha = 0$. From the basic definitions (8) and (9), no field or currents then exist, and it is easily confirmed that
$$\overline{F}(\zeta,s)|_{\alpha=0} = is^{-1}\,e^{-(s+2i)^{\frac{1}{2}}\zeta}. \tag{37}$$

The inversion is conventional (see, for example, Campbell & Foster 1948, (819)):

$$F(\zeta,\tau)|_{\alpha=0} = \tfrac{1}{2}i\left\{e^{-(2i)^{\frac{1}{2}}\zeta}\operatorname{erfc}\left[\frac{\zeta}{2\tau^{\frac{1}{2}}}-(2i\tau)^{\frac{1}{2}}\right]+e^{(2i)^{\frac{1}{2}}\zeta}\operatorname{erfc}\left[\frac{\zeta}{2\tau^{\frac{1}{2}}}+(2i\tau)^{\frac{1}{2}}\right]\right\}. \tag{38}$$

This formula correctly describes the growth of an ordinary linear Ekman boundary layer (see Greenspan 1968, where the imaginary part of $F$ can also be seen plotted in figure 2.3). The important conclusion obtainable from this result is, that the steady Ekman layer (of thickness $\sim (\nu/\Omega)^{\frac{1}{2}}$) is approached through a series of inertial oscillations (at non-dimensional frequency 2) in a non-dimensional time of order $\tau = 2$. This rather short duration of the transient is also typical for the establishment of non-linear Ekman layers (e.g. Benton 1966), and has important implications for spin-up.

(ii) *Early time behaviour*

The early time behaviour of any of the functions in physical space is determined by the corresponding behaviour of the transform functions for large $|s|$. This fact is expressible as a precise limit by one of the well-known Tauberian theorems (e.g. Doetsch 1961), which states that

$$\lim_{\tau\to 0} F(\zeta,\tau) = \lim_{s\to\infty} s\bar{F}(\zeta,s), \tag{39}$$

provided the limits exist. It is straightforward, but tedious, to regard $\alpha$, $\delta$ and $\zeta$ as fixed, and to obtain what are in effect asymptotic expansions for $\bar{F}$, $\bar{M}$, $\bar{W}$, valid as $s \to \infty$. In this way it is found that

$$k \sim s^{\frac{1}{2}} + \left(\frac{\alpha^2}{1-\delta} + i\right) s^{-\frac{1}{2}} + O(s^{-\frac{3}{2}}), \tag{40}$$

$$m \sim (\delta s)^{\frac{1}{2}} - \frac{\delta^{\frac{1}{2}}\alpha^2}{1-\delta} s^{-\frac{1}{2}} + O(s^{-\frac{3}{2}}), \tag{41}$$

$$\begin{aligned}\bar{F}(\zeta,s) \sim \frac{i}{s}\left\{1 - \frac{[\alpha^2 + (1-\delta)\,i]\,\zeta}{(1-\delta)\,s^{\frac{1}{2}}} + O(s^{-1})\right\} e^{-s^{\frac{1}{2}}\zeta} \\ + \frac{2i\delta^{\frac{1}{2}}\alpha^2}{(1-\delta)^2 s^2}\left[1 + \frac{\delta^{\frac{1}{2}}\alpha^2\zeta}{(1-\delta)\,s^{\frac{1}{2}}} + O(s^{-1})\right] e^{-(\delta s)^{\frac{1}{2}}\zeta}.\end{aligned} \tag{42}$$

The corresponding results for $\bar{M}$, $\bar{W}$ are omitted in the interest of brevity. Expansions of the type (42) can now be inverted term by term to yield a small time expansion for the various functions in physical space. The form of these expansions is of a type studied in a non-linear, non-magnetic context by Benton (1966), and in the present problem it is algebraically simpler to obtain them directly in physical space. The appropriate expansions are:

$$U(\zeta,\tau) = \tau[U_1(\eta) + \tau U_2(\eta) + \tau^2 U_3(\eta) + \ldots], \tag{43}$$

$$V(\zeta,\tau) = V_1(\eta) + \tau V_2(\eta) + \tau^2 V_3(\eta) + \ldots, \tag{44}$$

$$W(\zeta,\tau) = \tau^{\frac{3}{2}}[W_1(\eta) + \tau W_2(\eta) + \tau^2 W_3(\eta) + \ldots], \tag{45}$$

$$Q(\zeta,\tau) = \tau[Q_1(\eta) + \tau Q_2(\eta) + \tau^2 Q_3(\eta) + \ldots], \tag{46}$$

$$A(\zeta,\tau) = \tau^{\frac{3}{2}}[A_1(\eta) + \tau A_2(\eta) + \tau^2 A_3(\eta) + \ldots], \tag{47}$$

$$B(\zeta,\tau) = \tau^{\frac{1}{2}}[B_1(\eta) + \tau B_2(\eta) + \tau^2 B_3(\eta) + \ldots], \tag{48}$$

$$C(\zeta,\tau) = \tau^2[C_1(\eta) + \tau C_2(\eta) + \tau^2 C_3(\eta) + \ldots], \tag{49}$$

where

$$\eta = \frac{z}{2(\nu t)^{\frac{1}{2}}} = \frac{\zeta}{2\tau^{\frac{1}{2}}} \tag{50}$$

is the familiar similarity variable of viscous boundary-layer theory. If these formal series are substituted into (11)–(18), and coefficients of like powers of $\tau^{\frac{1}{2}}$

are equated to zero, a sequential hierarchy of ordinary differential equations emerge, which are straightforward (but laborious) to solve. It is found, for example, that $V_1(\eta) = \operatorname{erfc}\eta$, as in the ordinary Rayleigh problem. More generally, the solutions involve functions of both $\eta$ and $\delta^{\frac{1}{2}}\eta = z/2(\lambda t)^{\frac{1}{2}}$. Consequently, an important deduction is, that the $k$ layer starts off as a pure viscous diffusion region of thickness $(\nu t)^{\frac{1}{2}}$, whereas the $m$ layer is a purely magnetic or resistive diffusion layer of thickness $(\lambda t)^{\frac{1}{2}}$. This also follows from an inspection of (40)–(42).

The real value of the preceding expansions, and the hierarchy of equations which determines the coefficient functions, is not only that they provide a formally exact framework for expressing the small time solution, but moreover that a wealth of information can be obtained about the way in which this complicated flow begins *without extracting detailed analytical solutions*. Thus, we spare the reader mathematical details and simply describe the sequence of processes which successively come into play during the early stages of flow development.

Immediately following the impulse (i.e. to order $\tau^0$ in the expansions), a simple Rayleigh shear layer forms in the azimuthal flow; to this order in time, it is unaffected by either rotation or the magnetic field; its dimensional thickness is of order $(\nu t)^{\frac{1}{2}}$. Next, when the order $\tau^{\frac{1}{2}}$ terms first become important, the applied axial magnetic field is tipped by the Rayleigh shear into the azimuthal direction (thereby generating axial and radial electric currents); this azimuthal field has variations on two length scales, $(\nu t)^{\frac{1}{2}}$ and $(\lambda t)^{\frac{1}{2}}$. At order $\tau^1$, the pressure imbalance arising from centripetal acceleration associated with the azimuthal flow drives a radial motion; simultaneously, the simple Rayleigh shear layer is altered by the electromagnetic body force (the radial current crossed into the axial field gives an azimuthal body force). The important new effects, which enter at order $\tau^{\frac{3}{2}}$, are the generation of axial (Ekman) flow to balance the radial mass motion, and a radial field (or tangential current), induced by radial velocity tipping the axial magnetic field. Finally, at times of order $\tau^2$, the axial velocity stretches the imposed field lines, thereby increasing $B_z$.

The convergence of this type of series is known to be rapid in the non-magnetic case (Benton 1966), but presumably it cannot reveal much about inertial oscillations, and is also laborious to calculate; this approach will not be pursued further.

### (iii) *The steady-state solution and its approach*

Whereas the early time development is governed by large $|s|$, the final large time behaviour arises from regions of the complex plane near $s = 0$. If the steady-state limit exists, then it is given by (Doetsch 1961):

$$F(\zeta, \infty) = \lim_{s\to 0} s\bar{F}(\zeta, s).$$

Equations (32)–(35) show that as $s \to 0$ then $n \to \alpha^2 + i$, $q \to 0$, so

$$k \to (2\alpha^2 + 2i)^{\frac{1}{2}} = \beta + i\gamma, \quad m \to 0, \tag{51}$$

where $$\beta(\alpha) = [(1+\alpha^4)^{\frac{1}{2}} + \alpha^2]^{\frac{1}{2}}, \quad \gamma(\alpha) = [(1+\alpha^4)^{\frac{1}{2}} - \alpha^2]^{\frac{1}{2}} = \beta^{-1}. \tag{52}$$

The notation in (51) and (52) is identical to that used in Gilman & Benton. In particular, $\beta(0) = \gamma(0) = 1$, while for large $\alpha$,

$$\beta \sim \sqrt{2}\alpha = \frac{d_E}{d_H}, \quad \gamma \sim \frac{1}{\sqrt{2}\alpha} \to 0.$$

Note that, since $m \to 0$, the *thickness of the m layer tends to infinity as time tends to infinity*. This is precisely the non-uniformity which gives the steady-state solution a single layer structure whereas, at all finite times, the unsteady problem clearly has a double-layer structure. The reason is simply that, whereas the $k$ layer remains of bounded thickness for all time, the $m$ layer continues diffusing out to spatial infinity, becoming more nearly spatially uniform as it does so. The steady-state solutions are:

$$F(\zeta, \infty) = i\, e^{-(\beta+i\gamma)\zeta}, \tag{53}$$

$$M(\zeta, \infty) = -\frac{i}{\beta+i\gamma}[1-e^{-(\beta+i\gamma)\zeta}], \tag{54}$$

which are independent of $\delta$, as in Gilman & Benton. (The reason is that $\delta$ enters only as a multiplier of an unsteady term; see (26).) It now follows that

$$U(\zeta, \infty) = e^{-\beta\zeta}\sin\gamma\zeta, \tag{55}$$

$$V(\zeta, \infty) = e^{-\beta\zeta}\cos\gamma\zeta, \tag{56}$$

$$W(\zeta, \infty) = -\frac{2}{\beta^2+\gamma^2}[\gamma - e^{-\beta\zeta}(\gamma\cos\gamma\zeta + \beta\sin\gamma\zeta)], \tag{57}$$

$$A(\zeta, \infty) = -\frac{1}{\beta^2+\gamma^2}[\gamma - e^{-\beta\zeta}(\gamma\cos\gamma\zeta + \beta\sin\gamma\zeta)], \tag{58}$$

$$B(\zeta, \infty) = -\frac{1}{\beta^2+\gamma^2}[\beta - e^{-\beta\zeta}(\beta\cos\gamma\zeta - \gamma\sin\gamma\zeta)]. \tag{59}$$

If it is noticed that the present Rossby number differs in sign from that introduced by Gilman & Benton, then it is readily verified that the steady velocity field and electric current here agree exactly with theirs. This shows, as stated previously in § 2, that the fundamental fields are insensitive to some boundary conditions. Actually, the radial magnetic field differs by a constant from the value in Gilman & Benton. More importantly (58) and (59) show that neither $A$ nor $B$ decay to zero as $\zeta \to \infty$. This is, again, evidence of the non-uniform approach to the steady-state, the order of taking limits ($\tau \to \infty$, $\zeta \to \infty$) being crucial. These non-decaying constants are directly traceable to the quantity 1 in the square bracket of (54), which is the $m$-layer term. In effect, non-zero values of $A$, $B$ exist throughout the $m$ layer; when that layer diffuses all the way out to spatial infinity, as it must do in order for the steady state to be approached, it carries these non-decaying values with it, thereby changing some boundary conditions at infinity. The appropriate order for taking these limits, which preserves all the boundary conditions imposed in (22), is $\zeta \to \infty$, $\tau \to \infty$, rather than the reverse. In the problem of ultimate interest, the fluid will, of course, be confined to a finite spatial region between two plates, and the spin-up will take place within a finite time; the non-uniform limit of $\zeta$ and $\tau$, both tending to infinity, thus loses relevance.

When the results in Gilman & Benton are now brought to bear, we can summarize the ultimate state, to which the present flow evolves, as consisting of a single Ekman–Hartmann boundary layer near the plate, whose important features are an Ekman suction outside the layer reduced from the classical non-magnetic value, and simultaneously an induced axial electric current (the Hartmann current), which persists unabated to spatial infinity. The thickness of the layer is clearly of order $d_E/\beta$; hence, for $\alpha = 0$, we have an Ekman layer, while for large $\alpha$, we obtain (since $\beta \sim d_E/d_H$) a Hartmann layer (which is thinner than the Ekman layer). Henceforth, the $k$ layer is referred to by the more revealing name of Ekman–Hartmann boundary layer. It is clearly one fundamental type of structure in hydromagnetic spin-up problems. The fact that its growth, initially by viscous diffusion, eventually slows down and reaches an asymptotic bounded steady state is due to the balancing effect, not of convection as in ordinary boundary layers, but rather to the distortion of vortex lines and magnetic field lines in the region exterior to it. Similar reasoning (which follows) also explains why the $m$ layer continues to diffuse without limit. The fundamental transform solutions (30), (31) and (36) show that at any large finite time (non-zero s), the state of affairs outside the $m$ layer (when it is still of finite but large thickness) consists only of an Ekman suction velocity. There is, consequently, no distortion of either vortex or magnetic field lines there (in particular the axial Hartmann electric current immediately outside the Ekman–Hartmann layer must eventually turn completely into the radial direction within the $m$ layer, since it falls to zero outside the $m$ layer; this is an important consequence of the fact that all magnetic field perturbations vanish as $\zeta \to \infty$ when $\tau$ is finite). Consequently, there is no force outside the $m$ layer to balance its magnetic diffusion; it grows in thickness without limit.

### (iv) *Transient dynamics in the limit of vanishing magnetic Prandtl number* ($\delta = 0$)

For the physical systems of interest (earth, sun, laboratory fluids) the magnetic Prandtl number $\delta$ is much smaller than one. In this section, the exact inversions are found for the limit $\delta \to 0$, but it must be emphasized at the outset that this limit is a singular one ((26) shows that since $\delta$ multiplies the time derivative term, it produces a temporal singular perturbation). The non-uniformity is clarified by the approximate analysis of § 5, which examines the situation when $0 < \delta \ll 1$. In essence, the case $\delta = 0$ reveals the transient dynamics of the Ekman–Hartmann layer (the $k$ layer), but nothing is learned about the $m$ layer in this limit.

Letting $\delta$ tend to zero in (32) to (35) shows that

$$n \to \tfrac{1}{2}(s + 2\alpha^2 + 2i), \quad q \to 0,$$

whence

$$k \to (s + 2\alpha^2 + 2i)^{\frac{1}{2}}, \quad m \to 0. \tag{60}$$

The fact that $m \to 0$ explains why the $m$ layer dynamics are suppressed in the limit $\delta \to 0$; the layer loses its structure in this limit, just as it does in the limit $\tau \to \infty$ for $\delta \neq 0$. Another way of saying this is, when $\delta = 0$, immediately following the impulse, the $m$ layer diffuses instantaneously out to infinity and becomes spatially uniform; what remains is the important Ekman–Hartmann layer. Its detailed time dependent development is found by substituting (60) into (30), (31) and (36), which gives

$$\bar{F}(\zeta, s)|_{\delta=0} = is^{-1}e^{-(s+2\alpha^2+2i)^{\frac{1}{2}}\zeta}, \tag{61}$$

$$\bar{M}(\zeta, s)|_{\delta=0} = -is^{-1}(s+2\alpha^2+2i)^{-\frac{1}{2}}[1-e^{-(s+2\alpha^2+2i)^{\frac{1}{2}}\zeta}], \tag{62}$$

$$\bar{W}(\zeta, s)|_{\delta=0} = \mathrm{Re}\{-2is^{-1}(s+2\alpha^2+2i)^{-\frac{1}{2}}[1-e^{-(s+2\alpha^2+2i)^{\frac{1}{2}}\zeta}]\}. \tag{63}$$

Note that, in this limit, as previously mentioned, there is a branch point at $s = -2\alpha^2-2i$. Since the quantity in the braces in (63) differs only by a factor 2 from the expression for $\bar{M}$, we list the inversions of (61) and (62) only, which are (Campbell & Foster (546), (819) and (825)):

$$F(\zeta, \tau)|_{\delta=0} = \frac{i}{2}\left\{e^{-(\beta+i\gamma)\zeta}\operatorname{erfc}\left[\frac{\zeta}{2\tau^{\frac{1}{2}}}-(\beta+i\gamma)\tau^{\frac{1}{2}}\right]+e^{(\beta+i\gamma)\zeta}\operatorname{erfc}\left[\frac{\zeta}{2\tau^{\frac{1}{2}}}+(\beta+i\gamma)\tau^{\frac{1}{2}}\right]\right\}, \tag{64}$$

$$\begin{aligned} M(\zeta, \tau)|_{\delta=0} = &-(\beta+i\gamma)^{-1}\operatorname{erf}[(\beta+i\gamma)\tau^{\frac{1}{2}}] \\ &+\frac{i}{2(\beta+i\gamma)}\left\{e^{-(\beta+i\gamma)\zeta}\operatorname{erfc}\left[\frac{\zeta}{2\tau^{\frac{1}{2}}}-(\beta+i\gamma)\tau^{\frac{1}{2}}\right]-e^{(\beta+i\gamma)\zeta}\operatorname{erfc}\left[\frac{\zeta}{2\tau^{\frac{1}{2}}}+(\beta+i\gamma)\tau^{\frac{1}{2}}\right]\right\}. \end{aligned} \tag{65}$$

It may help the reader in what follows to point out that, despite its complexity, (64) is very much like the corresponding non-magnetic result in (38). (Compare, (37) and (66).)

The small time behaviour of (64) and (65) is of the type described by the series (43)–(49). To see this substitute that $\zeta = 2\tau^{\frac{1}{2}}\eta$ into (64) and (65), treat $\eta$ as fixed and expand the various functions in powers of $\tau^{\frac{1}{2}}$. The resulting coefficients will, in the present case ($\delta = 0$), be functions only of $\eta$, which is the fundamental similarity variable for the Ekman–Hartmann layer and, of course, $\alpha$; there is no dependence on $\delta^{\frac{1}{2}}\eta$, which is the $m$-layer variable. The large time behaviour is found by using the asymptotic expansion for the complementary and ordinary error functions. The results can be written as

$$F(\zeta, \tau)|_{\delta=0} \sim ie^{-(\beta+i\gamma)\zeta}-\frac{i\eta}{\pi^{\frac{1}{2}}(2\alpha^2+2i)\tau}e^{-\eta^2-(2\alpha^2+2i)\tau}+O(\tau^{-2}e^{-\eta^2-(2\alpha^2+2i)\tau}), \tag{66}$$

$$\begin{aligned} M(\zeta, \tau)|_{\delta=0} \sim &-i(\beta+i\gamma)^{-1}\left[1-e^{-(\beta+i\gamma)\zeta}-\frac{1}{\pi^{\frac{1}{2}}(\beta+i\gamma)\tau^{\frac{1}{2}}}e^{-(2\alpha^2+2i)\tau}\right] \\ &-\frac{i}{\pi^{\frac{1}{2}}(2\alpha^2+2i)\tau^{\frac{1}{2}}}e^{-\eta^2-(2\alpha^2+2i)\tau}+O(\tau^{-\frac{3}{2}}e^{-(2\alpha^2+2i)\tau}). \end{aligned} \tag{67}$$

These formulae are not difficult to interpret and contain some important results. In particular, in each expression the terms independent of $\tau$ (which are what the expansions reduce to when $\tau \to \infty$) agree with the steady-state solutions of (53) and (54). Consequently, to dominant order, the large time behaviour is, as expected, the steady Ekman–Hartmann boundary layer already described (it is unaffected by the value of $\delta$). The ultimate approach to this state is dominated by a relatively thick diffusively growing region (of thickness $(\nu t)^{\frac{1}{2}}$), in which small inertial oscillations are present. For non-zero magnetic interaction parameter $\alpha$, these inertial oscillations are exponentially damped in time (by basically the same mechanism that inhibits radial pumping and Ekman suction velocity, i.e. Maxwell tension in the applied field). In the non-magnetic case ($\alpha = 0$), (66) coincides with the asymptotic expansion of (38), and then the inertial oscillations are only algebraically damped in time. Notice that, even though the hydromagnetic interaction produces an exponential decay in the amplitude of these oscillations, it does not affect the thickness of the layer in which they exist. All of this pertains only to the region within the Ekman–Hartmann layer. It will subsequently be found that inertial oscillations in the generally much thicker $m$ layer are far more persistent in time.

Without question the most important features of the Ekman–Hartmann layer are the Ekman suction velocity and Hartmann electric current induced immediately outside. Since, in the limit $\delta \rightarrow 0$, the $m$ layer is spatially uniform

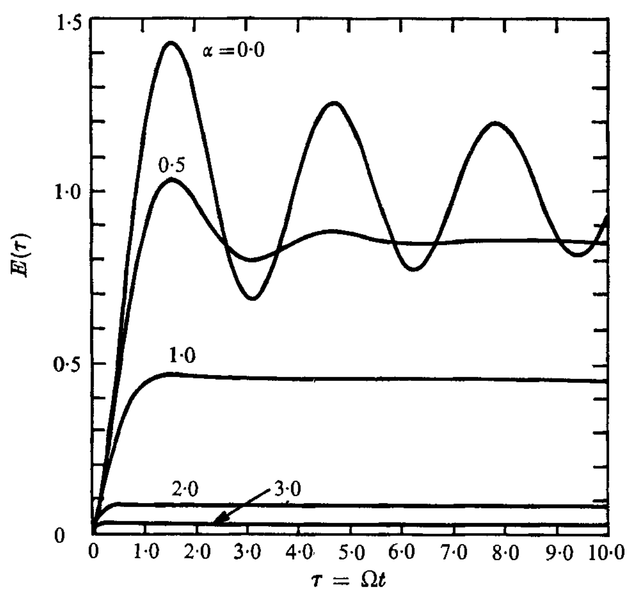


FIGURE 1. Normalized Ekman suction velocity at edge of Ekman–Hartmann boundary layer as a function of non-dimensional time for several values of magnetic interaction parameter.

and infinitely thick, the subsequent limit $\zeta \rightarrow \infty$ will represent a point still within the $m$ layer, but outside the Ekman–Hartmann layer (in § 5, we will examine these quantities outside both the $k$ and $m$ layers, in which case $\zeta$ must tend to infinity first). Denoting by subscript 1 values at the outer edge of the Ekman–Hartmann layer, we find, utilizing (6), (9), (10), (24), (62), (63):

$$[\overline{W}_1(s) + i\overline{K}_1(s)]_{\delta=0} = -2is^{-1}(s+2\alpha^2+2i)^{-\frac{1}{2}}; \quad (68)$$

and the inversion (Campbell & Foster (546)) is:

$$[W_1(\tau) + iK_1(\tau)]_{\delta=0} = -2i(\beta+i\gamma)^{-1}\,\mathrm{erf}\,[(\beta+i\gamma)\,\tau^{\frac{1}{2}}]. \quad (69)$$

Since the real and imaginary parts of this expression are both negative, we see that, when the boundary has been spun up ($\epsilon > 0$) and $B_0 > 0$, both the axial velocity and axial current flow *into* the Ekman–Hartmann layer (i.e. towards the boundary). Equation (69) also reveals again the basically diffusive nature of the Ekman–Hartmann layer, and the presence of inertial oscillations. A new piece of information obtainable from (69) is the total transient time $\tau_0$ for the development of the Ekman–Hartmann layer and its dynamically important Ekman suction and Hartmann current. Apart from the oscillations, which may

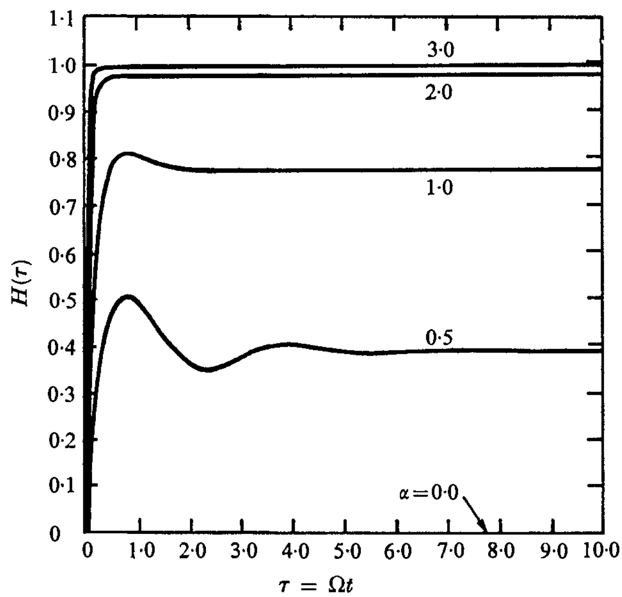


FIGURE 2. Normalized Hartmann electric current at edge of Ekman–Hartmann boundary layer as a function of a non-dimensional time for several values of magnetic interaction parameter.

persist longer, the error function will have approached its asymptotic value of 1 when the magnitude of its complex argument is about 2. Consequently,

$$\tau_0 = 4(\beta^2+\gamma^2)^{-1} = 2(1+\alpha^4)^{-\frac{1}{2}}. \tag{70}$$

For $\alpha = 0$, this reduces properly to the value $\tau_0 = 2$ appropriate to the establishment of an ordinary Ekman boundary layer; as the magnetic field increases, the Ekman–Hartmann layer develops even more rapidly (this result is, again, of decisive importance for spin-up).

The real and imaginary parts of (69) are plotted in figures 1 and 2 so that the detailed development of Ekman suction, Hartmann current, and inertial oscillations can be visualized. In these figures, the functions are normalized exactly as in Gilman & Benton (figure 5), i.e. Ekman suction and Hartmann current are divided by their steady-state values in the Ekman limit of $\alpha \to 0$ and Hartmann limit of $\alpha \to \infty$, respectively. Figure 1 shows that, apart from inertial oscillations, the Ekman suction is always less than the classical non-magnetic value.

## 5. The case of small finite magnetic Prandtl number ($0 < \delta \ll 1$)

The analysis of the preceding § 4 (iv), which is exact, has revealed the transient dynamics for the limiting case $\delta = 0$. In that relatively simple limit, the flow consists of only two regions, a transient Ekman–Hartmann boundary layer (the $k$ layer) and the spatially and temporally uniform region beyond it (the $m$ layer). Attention is now turned to the more realistic (but less tractable) situation in which the magnetic Prandtl number $\delta$, while still very much smaller than 1 (from physical considerations), is nonetheless greater than zero. In what follows, the plate is regarded as horizontal with the fluid lying above it.

The most obvious overall difference between the flow when $\delta = 0$ and that for $\delta > 0$ is that, in the latter case, the $m$ layer will have a finite thickness at any finite time after the impulsive start; thus, there must be a third region of the flow, that beyond both $k$ and $m$ layers. This outermost region is called the current-free region, for reasons to be seen soon. The main concern of this section is to investigate the dynamics of the $m$ layer and the current-free region. The exact transform solutions are mathematically intractable, so an approximate treatment will have to suffice. The desired goal is to find suitable approximate versions of $\bar{F}$, $\bar{M}$, which are valid on a legitimate inversion contour, and which are more tractable than the full expressions.

The generalized study of the initial transient processes (§ 4 (ii)) has shown that for $\tau > 0$, both $k$ and $m$ layers are present and the initial thicknesses are of order $(\nu t)^{\frac{1}{2}}$ and $(\lambda t)^{\frac{1}{2}}$, respectively. Consequently, if $\delta \ll 1$ as we have assumed, then the $m$ layer is initially much thicker than the $k$ layer. As time proceeds (§ 4 (iii)), both layers continue to grow, but the $k$ layer growth eventually stops, whereas the $m$ layer growth does not. Obviously, then, for the case of interest, the $m$ layer is *always* much thicker than the $k$ layer, so it should be relatively inviscid. The basis for an approximation valid within the $m$ layer could now be the inviscid approximation to $m$ itself, which is easily found from the inviscid version of (29) to be

$$m \doteq \left[\frac{\delta s(s+2i)}{s+2\alpha^2+2i}\right]^{\frac{1}{2}}. \tag{71}$$

Since this expression reduces to $(\delta s)^{\frac{1}{2}}$, and zero as $s$ tends to infinity and zero, exactly as the full expression for $m$ does, there is hope for the belief that at least the correct early and late time behaviour are preserved in this approximation. With an inviscid form for the $m$ layer, we could now, in principle, seek a uniformly valid approximation for the transform functions by the method of matched asymptotic expansions; however, matching in the complex $s$-plane is expected to involve subtle difficulties. Instead, a more direct, formal approach is adopted, which cannot be rigorously justified, but which does reduce properly to the exact cases studied above, and gives sensible results. (A method similar in spirit has been used with success by Greenspan & Howard in treating non-magnetic flow between two plates.)

Delaying the question of validity until later, we obtain the approximation formally by regarding $\delta$ as a small parameter compared to everything else; when the full expressions for $k$, $m$ (32)–(35) are expanded in powers of $\delta^{\frac{1}{2}}$, it is found that

$$\left.\begin{aligned} k &= (s+2\alpha^2+2i)^{\frac{1}{2}}[1+\phi_1(\delta)+O(\delta^2)] = k_0[1+\phi_1+O(\delta^2)],\\ m &= \left[\frac{\delta s(s+2i)}{s+2\alpha^2+2i}\right]^{\frac{1}{2}}[1-\phi_1(\delta)+O(\delta^2)] = m_0[1-\phi_1+O(\delta^2)], \end{aligned}\right\} \tag{72}$$

where

$$\phi_1(\delta) = \frac{\delta\alpha^2 s}{(s+2\alpha^2+2i)^2}. \tag{73}$$

These expansions are now substituted into the exponential and coefficient parts, separately, of (30) and (31), and the coefficient functions are expanded in powers of $\delta^{\frac{1}{2}}$. When terms only through order $\delta^{\frac{1}{2}}$ are retained, the resulting expressions can be written as

$$\bar{F}(\zeta, s) \doteq is^{-1}[e^{-k_0\zeta}-\phi_2(\delta)\,(e^{-k_0\zeta}-e^{-m_0\zeta})], \tag{74}$$

$$\bar{M}(\zeta, s) \doteq is^{-1}(s+2\alpha^2+2i)^{-\frac{1}{2}}[1-\phi_2(\delta)]\,(e^{-k_0\zeta}-e^{-m_0\zeta}), \tag{75}$$

where

$$\phi_2(\delta) = \frac{2\delta^{\frac{1}{2}}\alpha^2 s^{\frac{1}{2}}}{(s+2i)^{\frac{1}{2}}(s+2\alpha^2+2i)}. \tag{76}$$

Even though the completely formal procedure used to generate these expressions has no *a priori* justification, the resulting formulae appear to constitute a useful approximation with a considerable domain of validity. To begin with, notice that, from (72), the terms $k_0$, $m_0$ are such that their product is $(\delta s)^{\frac{1}{2}}(s+2i)^{\frac{1}{2}} = q$, exactly as is the product of the full $k, m$. More importantly, $m_0$ is just the inviscid approximation (71). Furthermore, the approximate functions in (74) and (75) are such that the imposed boundary conditions (at both $\zeta = 0$ and $\zeta = \infty$) are satisfied exactly, as is the initial condition. Also, these functions tend to precisely the same steady state as the exact functions (§ 4 (iii)). Finally, both ordinary hydrodynamic flow ($\alpha = 0$), and the zero magnetic Prandtl number case ($\delta = 0$), are recovered from (74) and (75), when $\alpha \to 0$ and $\delta \to 0$, respectively.

To delineate those regions of physical and parameter space (if any) where the approximations are not valid, it is necessary to ask whether an appropriate contour for the inversion integrals exists, everywhere on which (74) and (75) are close in some sense to the exact transform functions. If a contour can be found, on which both $\phi_1(\delta)$ and $\phi_2(\delta)$ are of small magnitude compared to 1, then presumably the truncated expansions (74) and (75) should be valid approximations. Showing that such contours exist is made somewhat easy by the following fortunate features of the 'expansion parameters' $\phi_1, \phi_2$: apart from the fact that they both vanish when $\delta \to 0$, they are also zero in the limits, $s \to 0$, $s \to \infty$ and $\alpha^2 \to 0$ ($\phi_1$, but not $\phi_2$ tends to zero as well when $\alpha^2 \to \infty$). Only when $s \to -2i$, or $-2\alpha^2-2i$, do $\phi_1$ or $\phi_2$ diverge, so clearly the contour must avoid these singularities by some, as yet unknown, amount. The immediate suspicion is that the approximations may not be adequate for describing the inertial oscillations, but should be legitimate for everything else.

With the given singularities at $s = 0, -2i, -2\alpha^2-2i$, the fundamental contour is a vertical straight line anywhere in the right half-plane. If all branch cuts are drawn horizontally to the left, then the basic contour can be deformed into the path shown in figure 3. As in Greenspan & Howard, the rays approaching infinity guarantee exponentially rapid convergence of the inversion integrals for large $|s|$; since $|\phi_1|$, $|\phi_2|$ are moreover small in that limit, clearly the early time behaviour is quite adequately described by (74) and (75). The limit of validity for the large time behaviour is governed (in inverse manner) by the extent to which the contour must indent into the right half-plane. Since $\phi_1$, $\phi_2 \to 0$ as $s \to 0$, the circular arc near the origin in figure 3 need only have infinitesimal radius. In fact,

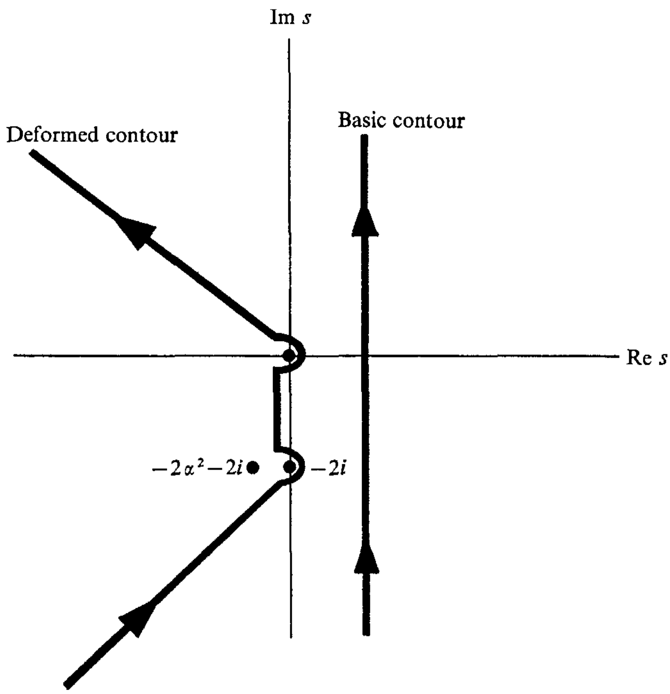


FIGURE 3. Contour for the inversion integrals.

only in the immediate neighbourhood of $s = -2i$ must a finite positive real part for $s$ be tolerated. Detailed examination of $\phi_1$, $\phi_2$ for $s$ near $-2i$ shows that the sensitive situation occurs if $|s+2i| \leqslant O(\delta)$. Then the approximation is not valid (in so far as the inertial oscillations are concerned) for $\tau \geqslant O(\delta^{-1})$. Generally, the approximation appears to be uniformly valid for all space, all $\alpha$ and

$$0 \leqslant \tau < O(\delta^{-1}).$$

(Since $\delta \ll 1$, the limitation in time does not exclude many of the transient processes of interest.)

While (74) and (75) are not yet exactly invertible, the dominant behaviour of the non-zero $\delta$ case can be extracted. In particular, suppose the $k$ layer terms of (74) are grouped together (i.e. those involving $\exp -k_0\zeta$). Then it is apparent that the leading term independent of $\delta$ is the same as (61) (so its inversion is (64)), but the next contribution involves $\delta^{\frac{1}{2}}$ only as a proportionality. (Similar remarks cannot be made about the $m$-layer terms, because $m_0$ itself depends upon $\delta$.) If (74) and (75) were inverted exactly, then it would be found that, for finite non-zero $\delta$, the basic ($\delta = 0$) $k$-layer dynamics undergo only a small correction of order $\delta^{\frac{1}{2}}$, no matter how large $\alpha$. (Also, the thickness of that layer remains unaffected to order $\delta^{\frac{1}{2}}$.) Consequently, the formula (64) should serve to describe adequately the growing Ekman–Hartmann layer in most physical situations where $\delta \ll 1$.

The more interesting aspect of (74) is the term, which gives the $m$-layer dynamics:

$$\bar{F}_m(\zeta, s) = \frac{2i\delta^{\frac{1}{2}}\alpha^2}{s^{\frac{1}{2}}(s+2i)^{\frac{1}{2}}(s+2\alpha^2+2i)} \exp\left(-\frac{(\delta s)^{\frac{1}{2}}(s+2i)^{\frac{1}{2}}}{(s+2\alpha^2+2i)^{\frac{1}{2}}}\zeta\right). \tag{77}$$

Because $\zeta$ occurs only in combination with $\delta^{\frac{1}{2}}$, and because $\alpha$ is independent of viscosity, we see that, as suggested previously, the $m$ layer is basically inviscid, depending instead only on the resistivity. Henceforth, it is referred to as the magnetic diffusion region; since it grows in thickness without limit, it would be misleading to call it a magnetic boundary layer.

Now (77) is a valid approximation for all $\alpha$ provided the contour does not approach to within order $\delta$ of the singular point $s = -2i$. It can therefore be further simplified for the two cases of interest, $\delta \ll 2\alpha^2 \ll 1$ and $2\alpha^2 \gg 1$, either by ignoring $2\alpha^2$ compared with $s+2i$, or vice versa. (Roughly speaking, these cases correspond to $2\alpha^2\tau \ll 1$ and $2\alpha^2\tau \gg 1$, respectively.) For the former case, (77) becomes:

$$\bar{F}_m(\zeta, s) = \frac{2i\delta^{\frac{1}{2}}\alpha^2}{s^{\frac{1}{2}}(s+2i)^{\frac{3}{2}}} e^{-(\delta s)^{\frac{1}{2}}\zeta}, \tag{78}$$

and the inversion is (Campbell & Foster (529), (807) and convolution):

$$F_m(\zeta, \tau) = \frac{4i}{\pi}\delta^{\frac{1}{2}}\alpha^2 e^{-2i\tau}\int_0^\tau \left(\frac{\tau}{\xi}-1\right)^{\frac{1}{2}} \exp\left(-\frac{\delta\zeta^2}{4\xi}+2i\xi\right) d\xi. \tag{79}$$

In the interests of brevity, the asymptotics of this integral will not be presented. Its main features are evident by inspection. When $2\alpha^2 \ll 1$, the magnetic diffusion region depends in a characteristically exponential fashion on the square of a magnetic diffusion variable $\eta_m = \delta^{\frac{1}{2}}\zeta/2\tau^{\frac{1}{2}} = z/2(\lambda t)^{\frac{1}{2}}$. The region diffuses parabolically with time at the resistive rate. Within the layer, there are inertial oscillations and weak horizontal velocities of order $\delta^{\frac{1}{2}}\alpha^2$.

From the definition of $\alpha$, (19), the case $2\alpha^2 \gg 1$ arises when the imposed magnetic field is strong and/or the rotation slow. In this situation the Alfvén mechanism can be anticipated to be important. Equation (77) reduces to:

$$\bar{F}_m(\zeta, s) \doteq \frac{i\delta^{\frac{1}{2}}}{s^{\frac{1}{2}}(s+2i)^{\frac{1}{2}}} \exp\left\{-\left(\frac{\delta}{2\alpha^2}\right)^{\frac{1}{2}} s^{\frac{1}{2}}(s+2i)^{\frac{1}{2}}\zeta\right\};$$

and the inversion is (Campbell & Foster (861)):

$$F_m(\zeta, \tau) \doteq \begin{cases} i\delta^{\frac{1}{2}} e^{-i\tau} J_0\left(\left[\tau^2 - \dfrac{\delta\zeta^2}{2\alpha^2}\right]^{\frac{1}{2}}\right), & \text{for } \left(\dfrac{\delta}{2\alpha^2}\right)^{\frac{1}{2}}\zeta < \tau, \\ 0 & \text{for } \left(\dfrac{\delta}{2\alpha^2}\right)^{\frac{1}{2}}\zeta > \tau. \end{cases} \tag{80}$$

Substitution of the fundamental definitions shows that the argument of the Bessel function here is indeed constant on surfaces which propagate with (dimensional) speed $B_0/(\rho\mu)^{\frac{1}{2}}$, the Alfvén speed. The front is sharp, but the 'tail' behind it displays a complicated oscillatory profile induced by rotation. The amplitude of the front (and the horizontal velocities within the magnetic diffusion region) are small, of order $\delta^{\frac{1}{2}}$, and this agrees with the non-rotating result (Shercliff 1965), being due to the fact that we have an insulating boundary, mechanically excited. (A stronger Alfvén front would be generated if the boundary were a conductor, or if the impulsive excitation were electrical.)

Without delving directly any further into the dynamics of the magnetic diffusion region, we turn now to the current-free region beyond it. In many ways, this region is as interesting and important for spin-up as the $m$ layer itself. To investigate the state of affairs outside both $k$ and $m$ layers, we regard $\delta$ as small but non-zero, and let $\zeta$ tend to infinity. Performing this operation directly on the exact transform solutions (30), (31) and (36) shows that $\bar{F}$ and $\bar{M}$ vanish, while $\bar{W}$ tends to

$$\bar{W}_2(s) \equiv \lim_{\zeta\to\infty} \bar{W}(\zeta, s) = \mathrm{Re}\left\{-2i\,\frac{k+m}{s(s+2i)^{\frac{1}{2}}\,[(s+2i)^{\frac{1}{2}}+(\delta s)^{\frac{1}{2}}]}\right\}. \tag{81}$$

Consequently, as mentioned previously, all electric currents, and the horizontal components of perturbation velocity, vanish outside the magnetic diffusion region; only an Ekman suction velocity remains.

Equation (81) cannot be simply inverted for general values of $\delta$ but, since $\delta$ can be arbitrarily small, we are tempted to examine the subsequent limit $\delta \to 0$. The simplified expression obtained in this way is:

$$\bar{W}_2(s)|_{\delta\to 0} = \mathrm{Re}\left[-2i\,\frac{(s+2\alpha^2+2i)^{\frac{1}{2}}}{s(s+2i)}\right]; \tag{82}$$

and the inversion (Campbell & Foster (549)) is:

$$W_2(\tau)|_{\delta\to 0} = \mathrm{Re}\{-(\beta+i\gamma)\,\mathrm{erf}\,[(\beta+i\gamma)\,\tau^{\frac{1}{2}}] + 2^{\frac{1}{2}}\alpha\, e^{-2i\tau}\,\mathrm{erf}\,[(2\alpha^2\tau)^{\frac{1}{2}}]\}. \tag{83}$$

The second term in the curly bracket here is an inertial oscillation (with fluid columns oscillating purely vertically), which does not damp out as time increases. Even though neither viscosity nor resistivity are operative in this outer most region, oscillations in vertical velocity must clearly be strongly linked to similar motions in the magnetic diffusion region, where damping does take place. This result must therefore be regarded suspiciously. A closer inspection shows that the non-decaying nature of this term is a result of taking the limit $\delta \to 0$; this changes the branch point at $s = -2i$ in the denominator of (81) into a simple pole (evident in (82)). A careful asymptotic analysis valid for small non-zero $\delta$, which is sketched in an appendix, shows that this oscillation does in fact damp out. (It damps out very slowly, however; this point is returned to below.) Furthermore, the first term in (83) is found to be accurate. As a result, for sufficiently large time, the Ekman suction velocity in the outer region is:

$$\lim_{\tau\to\infty} W_2(\tau) = -\beta. \tag{84}$$

When it is recalled (52) that for all non-zero $\alpha$, $\beta$ is greater than 1, then (84) shows that *the Ekman suction velocity outside of both dissipative layers is enhanced compared with the classical non-magnetic value*. In contrast, the value at the outer edge of the Ekman–Hartmann layer, which lies beneath the magnetic diffusion region, is smaller than the classical value (see (69)). This seemingly curious conclusion is actually one of the most fascinating and significant to emerge; its explanation relies on a partial summary of some of our previous results.

Let us restrict attention to non-dimensional times greater than of order 2 and, of course, $0 < \delta \ll 1$; then the Ekman–Hartmann layer has become quasi-steady, and the magnetic diffusion region is much thicker and continually diffusing. At the outer edge ($\zeta \approx \beta^{-1}$) of the Ekman–Hartmann layer, the circumferential perturbation velocity has fallen to order $\delta^{\frac{1}{2}}$, and there is an axial Ekman suction towards the boundary reduced from the non-magnetic value, and a significant axial Hartmann electric current in the same sense (if $B_0 > 0$). The electric current flowing out of the magnetic diffusion region everywhere at the bottom must, by continuity, eventually turn entirely into the radial direction within that region, since no electric current flows into it from outside. This radially inward perturbation electric current interacts with the impressed axial magnetic field to produce an accelerating positive azimuthal body force in the magnetic diffusion region. This explains the origin of the positive perturbation swirl velocity of order $\delta^{\frac{1}{2}}$ that has been seen to exist within the $m$ layer. Unbalanced centrifugal force arising therefrom will (as in the ordinary Ekman layer) drive a weak radial outflow which is compensated by extra axial velocity flowing into the layer. This self-consistent picture is illustrated schematically in figure 4, which is a meridian projection of the lines which are tangent to the velocity field (solid curves) and electric current (dashed curves). In effect the Ekman–Hartmann boundary layer accommodates the impressed boundary conditions on velocity to the outer flow, while the magnetic diffusion region arises to form the necessary transition for the electric current, which the Ekman–Hartmann layer induces at its outer edge.

It but remains to discuss the highly persistent inertial oscillations in the current-free region. A careful asymptotic analysis (see appendix) valid for small $\delta$ and large $\tau$ shows that

$$W_2(\tau) \sim -\beta + \frac{\delta^{\frac{1}{2}}\alpha}{(2\pi\tau)^{\frac{1}{2}}} + \frac{\alpha}{(2\pi\delta\tau)^{\frac{1}{2}}}(\cos 2\tau + \sin 2\tau). \tag{85}$$

Consequently, the monotonic part of the steady state is approached fairly quickly (when $\tau \gg \delta\alpha^2$) but the inertial oscillations damp out only at very large times ($\tau \gg \alpha^2/\delta$). It is of interest to remark that inertial oscillations in Greenspan & Howard's problem (with viscosity as the only dissipative mechanism) damp as $\tau^{-\frac{1}{2}}$; the extra factor $\delta^{-\frac{1}{2}}$ here is undoubtedly a reflexion of the fact that the present damping occurs in a less effective resistive region, whose thickness is of order $\delta^{-\frac{1}{2}}$ thicker than the viscous region. Since the amplitude of this inertial oscillation is proportional to $\alpha$ for large $\alpha$, and vanishes when $\alpha$ does, these oscillations are presumably of hydromagnetic origin. This can be shown clearly by combining the inviscid version of (11)–(16) into a single equation for the Ekman suction, which is

$$W_{\tau\tau} + 4W = -8\alpha^2 B - 4\alpha^2 A_\tau, \tag{86}$$

where a constant of integration (more precisely a function of time) has been omitted. Naturally, a completely consistent solution of (86) can be obtained only by simultaneously solving for $A$ and $B$. However, since the gross properties of the exact solution are now known, we can, conceptually, treat the right-hand side of (86) as a given forcing function and infer qualitatively the behaviour of $W$. In the current-free region, since both $A$ and $B$ vanish (always), there is no direct forcing, and the only possibility for $W$ is undamped free inertial oscillations at frequency $2\Omega$. However, (86) ought to be a reasonable approximation within the magnetic diffusion region as well, because only the viscous contributions to it have been neglected. For large time, when the term $A_\tau$ is presumably small compared with $B$, we see that inertial oscillations can be driven by the Hartmann current function $B$, which has some non-zero average value throughout the magnetic diffusion region. Axial motions so forced can be expected to persist out into the current-free region, and in essence to be damped by resistivity within the magnetic diffusion region.

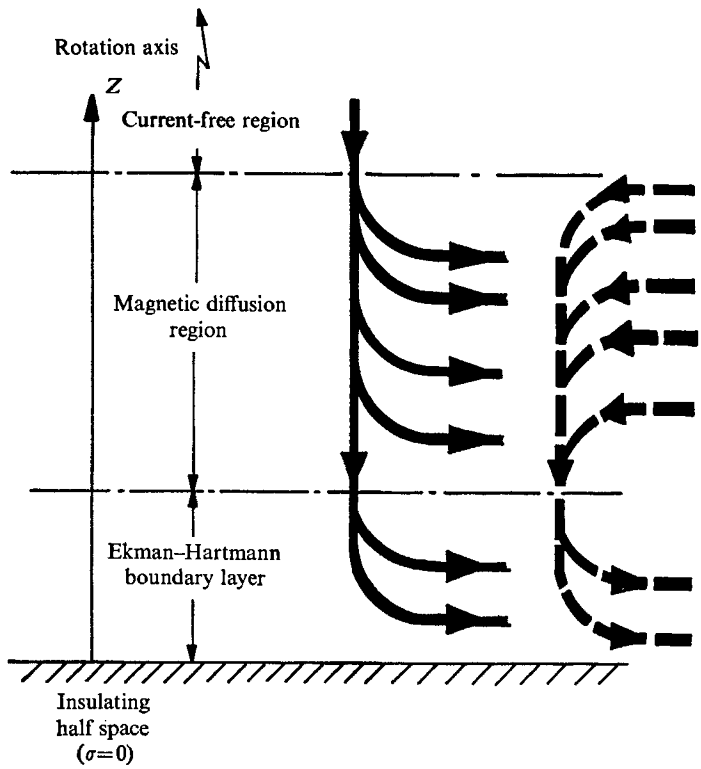


FIGURE 4. ——, Schematic meridional topology of lines tangent to velocity field; —·——·—, electric current.

## 6. Summary

The concern herein has been to understand the impulsively started transient approach to a linear, steady Ekman–Hartmann boundary layer on an infinite, flat, insulating plate in the presence of an imposed, uniform, axial, magnetic field. The plate angular velocity is considered to be increased from $\Omega$ to $\Omega(1+\epsilon)$, and $\epsilon \ll 1$.

The solution of this problem is dependent on two non-dimensional parameters: the magnetic Prandtl number $\delta$, which is the ratio of viscous to magnetic diffusivity and the hydromagnetic interaction parameter $\alpha$, which measures the strength of the electromagnetic body force ($\mathbf{j} \times \mathbf{B}$) relative to the Coriolis force. At all finite times after the impulse, and for non-zero but small $\delta$, and non-zero $\alpha$, the flow is found to consist of three distinct regions whose features are summarized as follows:

(i) Immediately adjacent to the boundary there is an Ekman–Hartmann boundary layer, which acts as a transition region for the change in velocity field required by the boundary conditions. It is the only region of the flow in which viscosity is crucial, but resistivity also acts therein. This layer begins growing like $(\nu t)^{\frac{1}{2}}$, and approaches the steady profile described by Gilman & Benton (1968) in a dimensional time of order $2(1+\alpha^4)^{-\frac{1}{2}}\,\Omega^{-1}$. The steady thickness is

$$[(1+\alpha^4)^{\frac{1}{2}}-\alpha^2]^{\frac{1}{2}}\,(\nu/\Omega)^{\frac{1}{2}},$$

which lies between that of the classical Ekman layer, $(\nu/\Omega)^{\frac{1}{2}}$ (which is precisely what the flow reduces to when $\alpha = 0$), and the smaller value appropriate to the pure Hartmann layer, which is $(\rho\mu\nu\lambda)^{\frac{1}{2}}/B_0$ (achieved when $\alpha \gg 1$). The significant features of this Ekman–Hartmann layer are that the efficiency of Ekman radial pumping is reduced by hydromagnetic interaction; therefore the Ekman suction is inhibited compared with the non-magnetic value, and simultaneously the imposed shear results in the inducement of an axial electric current, which flows into the layer if $B_0 > 0$. Inertial oscillations at frequency $2\Omega$ within this layer damp out rapidly in time (as $\exp(-2\alpha^2\Omega t)$) because of the Maxwell tension in the imposed magnetic field.

(ii) If the plate boundary is taken as horizontal with the fluid lying over it, then immediately above the Ekman–Hartmann layer is an inviscid magnetic diffusion region. This region arises to satisfy the exterior boundary conditions on electric current, which the Ekman–Hartmann layer is incapable of doing. For $\alpha^2 \ll 1$ it continuously grows parabolically in time by resistive diffusion, ultimately becoming infinitely thick and spatially uniform. At all times, it is much thicker than the Ekman–Hartmann layer. The Hartmann electric current leaving the magnetic diffusion region at the bottom is supplied by radial currents within; these in turn couple with the impressed axial field to produce an accelerating tangential electromagnetic body force, which acts to spin up very slightly (to order $\delta^{\frac{1}{2}}\alpha^2$) the fluid in the magnetic diffusion region. The excess centrifugal forces so produced lead to a weak radial outflow in this region, which is then supplied by an increased axial inflow from the current-free region beyond. Small inertial oscillations in the magnetic diffusion region are damped slowly by electric resistivity. When $\alpha^2 \gg 1$, the outer edge of the magnetic diffusion region no longer simply diffuses away from the plate, but instead becomes a weak Alfvén front (of strength $\delta^{\frac{1}{2}}$), and the tail is strongly modified by the fluid rotation.

(iii) In the current-free region adjacent to spatial infinity, the only non-zero perturbation is the Ekman suction velocity discussed above whose value is ultimately greater (by a factor $[(1+\alpha^4)^{\frac{1}{2}}+\alpha^2]^{\frac{1}{2}}$) than in the non-magnetic case. Completely undamped inertial oscillations in this Ekman suction constitute an exact mathematical solution to the inviscid equations in this region; but the actual state of affairs is that oscillations are driven hydromagnetically by similar but more complex motions within the magnetic diffusion region and these do damp out with time (but very slowly, as $\alpha(\delta\Omega t)^{-\frac{1}{2}}$).

The most important features for spin-up (which is treated in part 2 of this paper) are the conditions on axial velocity and axial current at the outer edge of the Ekman–Hartmann boundary layer and the outer edge of the magnetic diffusion region. For the former, Ekman suction is reduced (relative to non-conducting flow), but a Hartmann current is induced; for the latter, Ekman suction is enhanced and Hartmann current reduced (to zero). In part 2, we show that two different hydromagnetic spin-up mechanisms exist (both of which give the same rapid spin-up time), because of these two possibilities. In one, the spin-up is due entirely to the hydromagnetically enhanced Ekman suction, for the other, spin-up occurs, partly because of an electromagnetic body force arising from the Hartmann current.

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## Appendix. Asymptotic analysis of the Ekman suction velocity outside the magnetic diffusion region

The purpose of this appendix is to examine carefully the large time asymptotic behaviour of the Ekman suction velocity in the current-free region. In particular, we wish to show that the inertial oscillations do indeed damp out if the magnetic Prandtl number is non-zero.

The starting point is (81), which states that

$$\overline{W}_2(s) = \mathrm{Re}\left\{-2i\frac{k+m}{s(s+2i)^{\frac{1}{2}}\,[(s+2i)^{\frac{1}{2}}+(\delta s)^{\frac{1}{2}}]}\right\}. \tag{A 1}$$

Our interest is in the large time behaviour of the inverse transform when $\delta$ is small but non-zero. First, the numerator and denominator are multiplied by $k$, and the exact relationship (from (32)–(35)), $km = q = (\delta s)^{\frac{1}{2}}(s+2i)^{\frac{1}{2}}$, is substituted. With some further algebraic manipulation, (A 1) can be brought without approximation into the form,

$$\overline{W}_2(s) = \mathrm{Re}\,\{\overline{W}_{21}(s)+\overline{W}_{22}(s)+\overline{W}_{23}(s)\}, \tag{A 2}$$

where
$$\overline{W}_{21}(s) = -\frac{2i}{ks}, \quad (A\,3)$$

$$\overline{W}_{22}(s) = -\frac{2i(k^2-s-2i)}{ks[(1-\delta)s+2i)]}, \quad (A\,4)$$

$$\overline{W}_{23}(s) = +\frac{2i\delta^{\frac{1}{2}}(k^2-s-2i)}{ks^{\frac{1}{2}}(s+2i)^{\frac{1}{2}}[(1-\delta)s+2i]}. \quad (A\,5)$$

We now introduce the approximation (72), $k \doteq k_0 = (s+2\alpha^2+2i)^{\frac{1}{2}}$. For this to be valid, $|\phi_1| \ll 1$, which it is unless both $2\alpha^2 = O(\delta)$ and $\tau \geqslant O(\delta^{-1})$. In order to ensure validity for large time, we restrict interest, in this appendix, to values of $\alpha$ such that $2\alpha^2 \gg \delta$ (and of course $\delta \ll 1$). The other possible case ($2\alpha^2 \ll \delta$) is not of interest, because the oscillations will then be miniscule (their amplitude is proportional to $\alpha$.

With the above inequalities in mind, the inversions for $W_{21}$ and $W_{22}$ can then be expressed as (Campbell & Foster (210), (546)):

$$W_{21}(\tau) = -\frac{2i}{\beta+i\gamma}\operatorname{erf}[(\beta+i\gamma)\tau^{\frac{1}{2}}], \quad (A\,6)$$

$$W_{22}(\tau) = -\frac{4i\alpha^2}{(1-\delta)^{\frac{1}{2}}[2\alpha^2(1-\delta)-2i\delta]^{\frac{1}{2}}}\int_0^\tau \exp\left(-\frac{2i}{1-\delta}\xi\right)\operatorname{erf}\left[\left(2\alpha^2-\frac{2i\delta}{1-\delta}\right)^{\frac{1}{2}}\xi^{\frac{1}{2}}\right]d\xi. \quad (A\,7)$$

Integration by parts in (A 7) leads to:

$$W_{22}(\tau) = \frac{2\alpha^2(1-\delta)^{\frac{1}{2}}}{[2\alpha^2(1-\delta)-2i\delta]^{\frac{1}{2}}}\exp\left(-\frac{2i}{1-\delta}\tau\right)\operatorname{erf}\left[\left(2\alpha^2-\frac{2i\delta}{1-\delta}\right)^{\frac{1}{2}}\tau^{\frac{1}{2}}\right] - \frac{2\alpha^2}{\beta+i\gamma}\operatorname{erf}[(\beta+i\gamma)\tau^{\frac{1}{2}}]. \quad (A\,8)$$

With $k \doteq k_0$, the expression for $\overline{W}_{23}$ becomes

$$\overline{W}_{23}(s) = \frac{4i\delta^{\frac{1}{2}}\alpha^2}{1-\delta}\frac{1}{s^{\frac{1}{2}}(s+2i)^{\frac{1}{2}}(s+2\alpha^2+2i)^{\frac{1}{2}}\left(s+\dfrac{2i}{1-\delta}\right)}; \quad (A\,9)$$

and the inversion is (Campbell & Foster (546), (555) and convolution):

$$W_{23}(\tau) = \frac{4i\delta^{\frac{1}{2}}\alpha^2}{(1-\delta)[2\alpha^2(1-\delta)^{\frac{1}{2}}-2i\delta]^{\frac{1}{2}}}I(\tau), \quad (A\,10)$$

where
$$I(\tau) = \int_0^\tau \exp\left(-i\xi-\frac{2i}{1-\delta}(\tau-\xi)\right)J_0(\xi)\operatorname{erf}\left[\left(2\alpha^2-\frac{2i\delta}{1-\delta}\right)^{\frac{1}{2}}(\tau-\xi)^{\frac{1}{2}}\right]d\xi. \quad (A\,11)$$

Combination of (A 6), (A 8), (A 10) and (A 11) shows that

$$W_2(\tau) = \operatorname{Re}\left\{-(\beta+i\gamma)\operatorname{erf}[(\beta+i\gamma)\tau^{\frac{1}{2}}]+\frac{2\alpha^2(1-\delta)^{\frac{1}{2}}}{[2\alpha^2(1-\delta)-2i\delta]^{\frac{1}{2}}}\exp\left(-\frac{2i}{1-\delta}\tau\right)\right.$$
$$\times\operatorname{erf}\left[\left(2\alpha^2-\frac{2i\delta}{1-\delta}\right)^{\frac{1}{2}}\tau^{\frac{1}{2}}\right]+\frac{4i\delta^{\frac{1}{2}}\alpha^2}{(1-\delta)^{\frac{1}{2}}[2\alpha^2(1-\delta)-2i\delta]^{\frac{1}{2}}}\exp\left(-\frac{2i}{1-\delta}\tau\right)$$
$$\left.\times\int_0^\tau \exp\left(i\frac{1+\delta}{1-\delta}\xi\right)J_0(\xi)\operatorname{erf}\left[\left(2\alpha^2-\frac{2i\delta}{1-\delta}\right)^{\frac{1}{2}}(\tau-\xi)^{\frac{1}{2}}\right]d\xi\right\}. \quad (A\,12)$$

The only approximation used to obtain (A 12) is the replacement of $k$ by $k_0 = (s+2\alpha^2+2i)^{\frac{1}{2}}$. The first term here is, as stated in the main text, identical to the first term in (83), which gives the dominant long term monotonic behaviour. The other terms in (A 12) are the inertial oscillations. The integral can be written as

$$I(\tau) = I_1(\tau)+I_2(\tau)+I_3(\tau),$$

where
$$I_1(\tau) = \int_0^\infty \exp\left(i\frac{1+\delta}{1-\delta}\xi\right) J_0(\xi)\,d\xi,$$

$$I_2(\tau) = -\int_\tau^\infty \exp\left(i\frac{1+\delta}{1-\delta}\xi\right) J_0(\xi)\,d\xi,$$

$$I_3(\tau) = -\int_0^\tau \exp\left(i\frac{1+\delta}{1-\delta}\xi\right) J_0(\xi)\operatorname{erfc}\left[\left(2\alpha^2-\frac{2i\delta}{1-\delta}\right)^{\frac{1}{2}}(\tau-\xi)^{\frac{1}{2}}\right]d\xi.$$

The first integral here, when interpreted as a Laplace transform (Campbell & Foster (557)) is given by $i(1-\delta)/2\delta^{\frac{1}{2}}$. For $I_2(\tau)$, $\xi$ is large along the entire path of integration, so the asymptotic expansion of $J_0$ is used, which gives the following to dominant order:

$$I_2(\tau) \sim \frac{1+i}{4\pi^{\frac{1}{2}}}\frac{1-\delta}{\tau^{\frac{1}{2}}}\exp\left(\frac{2i}{1-\delta}\tau\right)+\frac{1-i}{4\pi^{\frac{1}{2}}}\frac{1-\delta}{\delta\tau^{\frac{1}{2}}}\exp\left(\frac{2i}{1-\delta}\delta\tau\right).$$

Inspection of $I_3$ shows it to be of order $1/\alpha^2\tau^{\frac{1}{2}}$, and therefore negligible compared with $I_2$ (since $\alpha^2 \gg \delta$); it will not be computed. Combination of terms gives

$$W_2(\tau) \sim \operatorname{Re}\left\{-(\beta+i\gamma)+\frac{1-i}{\pi^{\frac{1}{2}}}\frac{\delta^{\frac{1}{2}}\alpha^2}{\left(2\alpha^2-\dfrac{2i\delta}{1-\delta}\right)^{\frac{1}{2}}}\tau^{-\frac{1}{2}} +\frac{1+i}{\pi^{\frac{1}{2}}}\frac{\alpha^2}{\left(2\alpha^2-\dfrac{2i\delta}{1-\delta}\right)^{\frac{1}{2}}}(\delta\tau)^{-\frac{1}{2}}e^{-2i\tau}\right\}. \quad \text{(A 13)}$$

Neglecting the terms involving $2i\delta/(1-\delta)$ compared with $2\alpha^2$ leads finally to

$$W_2(\tau) \sim -\beta+\frac{\delta^{\frac{1}{2}}\alpha}{(2\pi\tau)^{\frac{1}{2}}}+\frac{\alpha}{(2\pi\delta\tau)^{\frac{1}{2}}}(\cos 2\tau+\sin 2\tau). \quad \text{(A 14)}$$

## REFERENCES

Barcilon, V. & Pedlosky, J. 1967 Linear theory of rotating stratified fluid motions. *J. Fluid Mech.* **29**, 1–16.

Benton, E. R. 1966 On the flow due to a rotating disk. *J. Fluid Mech.* **24**, 781–800.

Campbell, G. A. & Foster, R. N. 1948 *Fourier Integrals for Practical Applications.* New York: Van Nostrand.

Doetsch, G. 1961 *Guide to the Applications of Laplace Transforms.* London: Van Nostrand.

Gilman, P. A. & Benton, E. R. 1968 Influence of an axial magnetic field on the steady linear Ekman boundary layer. *Phys. Fluids*, **11**, 2397–401.

Greenspan, H. P. 1964 On the transient motion of a contained rotating fluid. *J. Fluid Mech.* **20**, 673–96.

Greenspan, H. P. 1965 On the general theory of contained rotating fluid motions. *J. Fluid Mech.* **22**, 449–62.

GREENSPAN, H. P. 1968 *The Theory of Rotating Fluids*. Cambridge University Press.

GREENSPAN, H. P. & HOWARD, L. N. 1963 On a time dependent motion of a rotating fluid. *J. Fluid Mech.* **17**, 385–404.

GREENSPAN, H. P. & WEINBAUM, S. 1965 On non-linear spin-up of a rotating fluid. *J. Math. Phys.* **44**, 66–85.

HOWARD, L. N., MOORE, D. W. & SPIEGEL, E. A. 1967 Solar spin-down problem. *Nature, Lond.* **214**, 1297–99.

PEDLOSKY, J. 1967 Spin-up of a stratified fluid. *J. Fluid Mech.* **28**, 463–80.

SHERCLIFF, J. A. 1965 *A Textbook of Magnetohydrodynamics*. London: Pergamon.